Chapter 1

WAKES AND IMPEDANCES

1.1 WAKE FIELDS

A positively charged particle at rest has static electric field going out radially in all directions. In motion with velocity v, magnetic field is generated. As the particle velocity approaches c, the velocity of light, the electric and magnetic fields are pancake-like, the electric field is radial and magnetic field azimuthal (the Liénard-Wiechert fields) with an open angle of about $1/\gamma$. It is worth pointing out that no matter how far away, this pancake is always perpendicular to the path of motion. In other words, the fields move with the test particle without any lagging behind as illustrated in Fig. 1.1. Such a field pattern is, of course, the steady-state solution of the problem.

When placed inside a perfectly conducting beam pipe, the pancake of fields is trimmed by the beam pipe. A ring of negative charges will be formed on the wall of the beam pipe where the electric field ends, and these image charges will travel at the same pace as the particle, creating the so called *image current*. If the wall of the beam pipe is not perfectly conducting or contains discontinuities, the movement of the image charges will be slowed down, thus leaving electromagnetic fields behind. For example, when coming across a cavity, the image current will flow into the wall of the cavity, creating fields trapped inside the cavity. These fields left behind by the particle are called *wake fields*, which are important because they will influence the motion of the particles that follow.

In addition to the wake fields, the electromagnetic fields seen by the beam particle

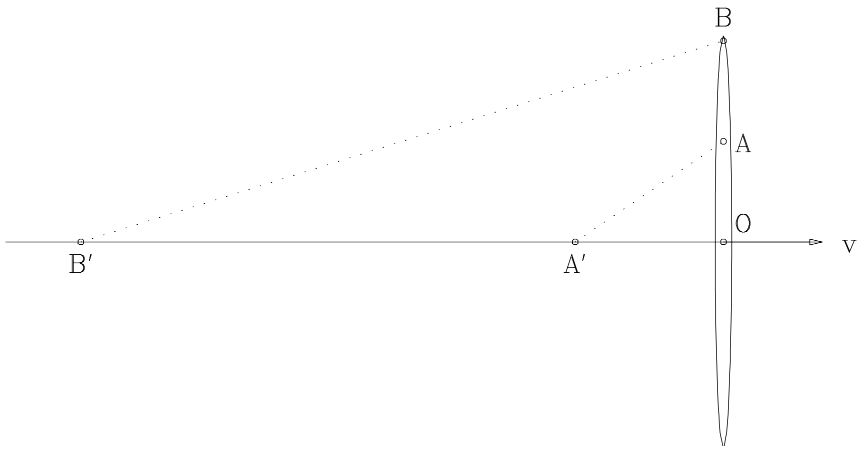

Figure 1.1: Schematic drawing of pan-cake electromagnetic fields emitted by an ultra-relativistic particle traveling with velocity v. The pan-cake is always perpendicular to the path of the particle and travels in pace with the particle no matter how far away the fields are from the particle. There is no violation of causality because fields at points A and B come from the particle at different locations. Fields from A are from A' at a time OA'/v ago, while fields at B from point B' at a time OB'/v ago.

consist of also the external fields from the magnets, rf, etc. We can write

$$(\vec{E},\vec{B})_{\substack{\text{seen by}\\ \text{particles}}} = (\vec{E},\vec{B})_{\substack{\text{external, from}\\ \text{magnets, rf, etc.}}} + (\vec{E},\vec{B})_{\substack{\text{wake}\\ \text{fields}}} \tag{1.1}$$

where

$$(\vec{E},\vec{B})_{\substack{\text{wake}\\ \text{fields}}} \left\{ \begin{array}{ll} \propto & N \quad \text{beam intensity} \\ \ll & (\vec{E},\vec{B})_{\text{external}} \end{array} \right.$$

Note that the last restriction, which is certainly not true in plasma physics, allows wake fields to be treated as perturbation. This perturbation, however, will break down when potential-well distortion is large. In that case, the potential-well distortion has to be included into the non-perturbative part. What we need to compute are the wake fields at a distance z behind the source particle and their effects on the test or witness particles that make up the beam. The computation of the wake fields is nontrivial. So approximations are required.

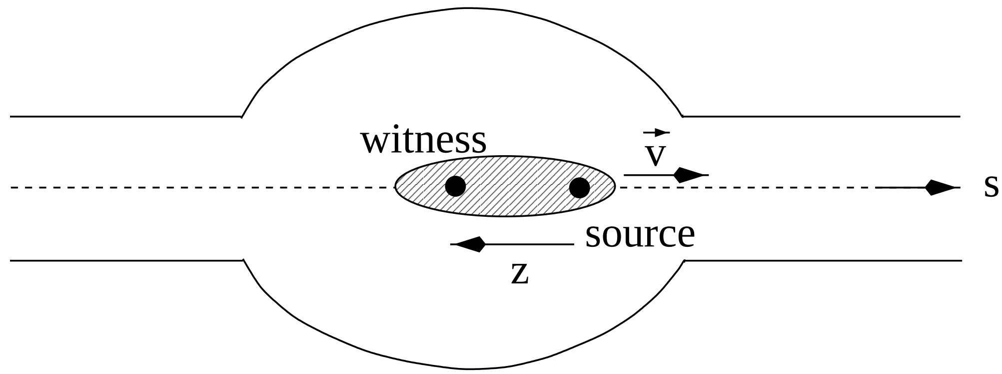

Figure 1.2: Schematic drawing of of a witness particle at a distance z behind the some source particle in a beam. Both particles are traveling along the direction s with velocity $\vec{v}$.

1.2 TWO APPROXIMATIONS

At high energies, the particle beam is rigid and the following two approximations apply:*

(1) **The rigid beam approximation**, which says that the beam traverses the discontinuity of the vacuum chamber rigidly and the wake field perturbation does not affect the motion of the beam during the traversal of the discontinuity. This implies that the distance z of the test particle behind some source particle as shown in Fig. 1.2 does not change.

(2) **The impulse approximation**. Although the test particle sees a wake force $\vec{F}$ coming from $(\vec{E}, \vec{B})$, what it cares is the impulse

$$\Delta\vec{p} = \int_{-\infty}^{\infty} dt\ \vec{F} = \int_{-\infty}^{\infty} dt\ e(\vec{E} + \vec{v} \times B) \tag{1.2}$$

as it completes the traversal through the discontinuity at its fixed velocity $\vec{v}$. Note that MKS units have been used in Eq. (1.2) and will be adopted throughout the rest of the lectures. We will therefore be coming across the electric permitivity of free space $\epsilon_0 = 10^7/(4\pi c^2)$ farads/m and the magnetic permeability of free space $\mu_0 = 4\pi \times 10^{-7}$ henry/m. These two quantities are related to the free-space impedance Z_0 and velocity of light c by

$$Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 2.99792458 \times 40\pi = 376.730313 \text{ Ohms} ,$$

*This approach to the Panofsky-Wenzel Theorem has been presented by A.W. Chao at OCPA Accelerator School, Hsinchu, Taiwan, August 3-12, 1998.

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 2.99792458 \times 10^8 \text{ m/s} . \tag{1.3}$$

Both $\vec{E}$, $\vec{B}$, and $\vec{F}$ are difficult to compute even at high beam energies. However, the impulse $\Delta\vec{p}$ has great simplifying properties through the Panofsky-Wenzel (P-W) Theorem, which forms the basis of wake potentials and impedances.

1.3 PANOFSKY-WENZEL THEOREM

Maxwell equations for a particle in the beam are:

$$\left\{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \dfrac{\rho}{\epsilon_0} & \text{Gauss's Law} \\ \vec{\nabla}\times\vec{B} - \dfrac{1}{c^2}\dfrac{\partial\vec{E}}{\partial t} = \mu_0\beta c\rho\hat{s} & \text{Ampere's Law} \\ \vec{\nabla}\cdot\vec{B} = 0 & \text{Gauss's Law for magnetic charge} \\ \vec{\nabla}\times\vec{E} + \dfrac{1}{c}\dfrac{\partial\vec{B}}{\partial t} = 0 & \text{Faraday's \& Lentz Law} \end{array}\right. \tag{1.4}$$

We have replaced the current density with $\vec{j} = \beta c\rho\hat{s}$ with $|\vec{v}| = \beta c$ and β will be treated as a constant, which is the result of the rigid-beam approximation, and is certainly true at high energies when $\beta \approx 1$. Note that we have been denoting the s-axis as the direction of motion of the beam, while reserving z as the distance the witness particle is behind the source particle. For a circular ring, the s-axis constitutes the axis of symmetry of the vacuum chamber. Together with the transverse coordinates x and y, they form an instantaneous Cartesian coordinate system. Thus, the above wake fields $\vec{E}$ and $\vec{B}$ as well as wake force $\vec{F}$ are function of x, y, s, t. From the rigid beam approximation, s is not independent, but is related to t by $s = z + \beta ct$, with z being time-independent. Since we are looking at the field behind a source, z is negative.

The Lorentz force on the test particle is $\vec{F} = e(\vec{E} + \beta c\hat{s} \times \vec{B})$. Here the rigid-beam approximation has also been used by requiring that the test particle has the same velocity as all other beam particles. It follows that

$$\begin{aligned} \vec{\nabla}\cdot\vec{F} &= \frac{4\pi e\rho}{\gamma^2} - \frac{e\beta}{c}\frac{\partial E_z}{\partial t} , \\ \vec{\nabla}\times\vec{F} &= -e\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial s}\right)\vec{B} . \end{aligned} \tag{1.5}$$

We are only interested in the impulse

$$\Delta\vec{p}(x,y,z) = \int_{-\infty}^{\infty} dt\ \vec{F}(x,y,z+\beta ct,t)\ ; \tag{1.6}$$

i.e., the integration of $\vec{F}$ along a rigid path with z being held fixed. Therefore

$$\vec{\nabla}\times\Delta\vec{p}(x,y,z) = \int_{-\infty}^{\infty} dt\ \left[\vec{\nabla}\times\vec{F}(x,y,s,t)\right]_{s=z+\beta ct}\ , \tag{1.7}$$

↑ this $\vec{\nabla}$ refers to x,y,z

↑ this $\vec{\nabla}$ refers to x,y,s

$$\begin{aligned}
\text{L.S.}\quad &= -e\int_{-\infty}^{\infty} dt\ \left[\left(\frac{\partial}{\partial t} + \beta c\frac{\partial}{\partial z}\right)\vec{B}(x,y,s,t)\right]_{s=z+\beta ct} \\
&= -e\int_{-\infty}^{\infty} dt\ \frac{d\vec{B}}{dt} = -e\vec{B}(x,y,z+\beta ct,t)\Big|_{t=-\infty}^{\infty} = 0\ .
\end{aligned}$$

Thus the P-W theorem reads

$$\boxed{\vec{\nabla}\times\Delta\vec{p} = 0\,.} \tag{1.8}$$

It is important to note that so far no boundary conditions have been imposed. The P-W theorem is valid for any boundaries ! The only needed inputs are the two approximations. Its even does not require $\beta = 1$. It just requires $\beta \approx 1$ so that β can remain constant Thus, the P-W theorem is very general.

The P-W theorem can be decomposed into a component parallel to the $\hat{s}$ and one perpendicular to $\hat{s}$:

$$\boxed{\vec{\nabla}\cdot(\hat{s}\times\Delta\vec{p}) = 0} \tag{1.9}$$

$$\boxed{\frac{\partial}{\partial z}\Delta\vec{p}_\perp = \vec{\nabla}_\perp\Delta p_s} \tag{1.10}$$

Equation (1.9) says something about the transverse components of $\Delta\vec{p}$, which becomes, in Cartesian coordinates,

$$\boxed{\frac{\partial\Delta p_x}{\partial y} = \frac{\partial\Delta p_y}{\partial x}} \tag{1.11}$$

On the other hand, Eq. (1.10) relates $\Delta\vec{p}_\perp$ and $\Delta\vec{p}_z$, that the transverse gradient of the longitudinal impulse is equal to the longitudinal gradient of the transverse impulse. Thus, the P-W theorem strongly constraints the components of $\Delta\vec{p}$.

There is an important supplement to the P-W theorem, which states:

$$\boxed{\beta = 1 \longrightarrow \vec{\nabla}_\perp\cdot\Delta\vec{p}_\perp = 0} \tag{1.12}$$

Proof:

$$\begin{aligned}
\vec{\nabla}\cdot\Delta\vec{p} &= \int_{-\infty}^{\infty} dt \left[\vec{\nabla}\cdot\vec{F}(x,y,s,t)\right]_{s=z+ct} = -\frac{e}{c}\int_{-\infty}^{\infty} dt \left[\frac{\partial E_s}{\partial t}\right]_{s=z+ct} \\
&= e\int_{-\infty}^{\infty} dt \left[\frac{\partial E_s}{\partial s}\right]_{s=z+ct} = \frac{\partial}{\partial s}\Delta p_s \ .
\end{aligned}$$

For the last step, since $s = z+ct$ is kept constant, therefore $\dfrac{\partial}{\partial t} = -c\dfrac{\partial}{\partial s}$. It is important to note that $4\pi e\rho/\gamma^2$, the first term of $\vec{\nabla}\cdot\vec{F}$ in Eq. (1.5) has been neglected in the proof.

1.4 CYLINDRICALLY SYMMETRIC BEAM CHAMBER

When the beam is inside a cylindrically symmetric vacuum chamber, naturally the cylindrical coordinates will be used. The P-W theorem, Eq. (1.8), and the supplemental theorem, Eq. (1.12), become [2]

$$\left\{\begin{aligned}
\frac{\partial}{\partial r}\left(r\Delta p_\theta\right) &= \frac{\partial}{\partial \theta}\Delta p_r \\
\frac{\partial}{\partial D}\Delta p_r &= \frac{\partial}{\partial r}\Delta p_s \\
\frac{\partial}{\partial D}\Delta p_\theta &= \frac{1}{r}\frac{\partial}{\partial \theta}\Delta p_s \\
\frac{\partial}{\partial r}\left(r\Delta p_r\right) &= -\frac{\partial}{\partial \theta}\Delta p_\theta \qquad (\beta = 1)
\end{aligned}\right. \tag{1.13}$$

Now, this set equations for $\Delta\vec{p}$ becomes surprisingly simple. It does not contain any source terms and is completely independent of boundaries, which can be conductors, resistive wall, dielectric, or even plasma. This result solely arrives from the Maxwell equations plus the two approximation.

There is no loss of generosity by letting $\Delta p_z \sim \cos m\theta$ with $m \geq 0$. Then, we get

$$\Delta p_s \sim \cos m\theta \quad \longrightarrow \quad \Delta p_r \sim \cos m\theta \quad \text{and} \quad \Delta p_\theta \sim \sin m\theta \ . \tag{1.14}$$

The set of equations for $\Delta\vec{p}$ becomes

$$\left\{ \begin{array}{rcl} \dfrac{\partial}{\partial r}(r\Delta p_\theta) & = & -m\Delta p_r \\ \dfrac{\partial}{\partial z}\Delta p_r & = & \dfrac{\partial}{\partial r}\Delta p_s \\ \dfrac{\partial}{\partial z}\Delta p_\theta & = & -\dfrac{m}{r}\Delta p_s \\ \dfrac{\partial}{\partial r}(r\Delta p_r) & = & -m\Delta p_\theta \qquad (\beta = 1) \end{array} \right. \tag{1.15}$$

From the first and last equations, we get, for $m \neq 0$,

$$\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}(r\Delta p_r)\right] = m^2\Delta p_r \ , \tag{1.16}$$

and therefore

$$\Delta p_r(r,\theta,z) \sim r^{m-1}\cos m\theta \ . \tag{1.17}$$

Now the whole solution can be written as

$$\left\{ \begin{array}{ll} c\Delta\vec{p}_\perp = -eI_m W_m(z) m r^{m-1}\left(\hat{r}\cos m\theta - \hat{\theta}\sin m\theta\right) & m \neq 0 \\ & \\ c\Delta p_s = -eI_m W_m'(z) r^m \cos m\theta & \text{all } m \end{array} \right. \tag{1.18}$$

In above, $W_m(z)$ is called the *transverse wake function* and $W_m'(z)$ the *longitudinal wake function*. They are related when $m \neq 0$ because of the P-W theorem. The wake functions are functions of one variable only, and are the only remaining unknown. They must be solved with boundary conditions. Recall that the complicated Maxwell-Vlasov equation that involves $\vec{E}$, $\vec{B}$, and sources has been reduced drastically to solving just for W_m.

More comments about Eq. (1.18). Here, e is the test charge and I_m is the electric mth multipole. Thus, W_m' has the dimension of force per charge square per length$^{(2m-1)}$ or Volts/Coulomb/m^{2m}, while W_m has the dimension of force per charge square per length^{2m} or Volts/Coulomb/m^{2m-1}. There are negative signs on the right sides to take care of the fact that the impulse is negative.

Recall that we have been looking at the wake force on a particle traveling at $s = z + vt$ behind a source particle traveling at $s = vt$. Thus $z < 0$. When $v \to c$, causality has to be imposed that $W_m(z) = 0$ when $z > 0$. For our discussions below, we will continue to use v instead of c in most places, because we would like to derive stability conditions and growth rates also for machines that are not ultra-relativistic. However, strict causality will be imposed.

Immediately behind a source particle, the test particle should receive a retarding force, otherwise conservation of energy will be violated. This implies that $W_m'(z) > 0$ when $|z|$ is small, recalling that the $W_m'(z)$ is defined in Eq. (1.18) with a negative sign on the right side. This is illustrated in Fig. 1.3. It will be proved later in Chapter 7 that

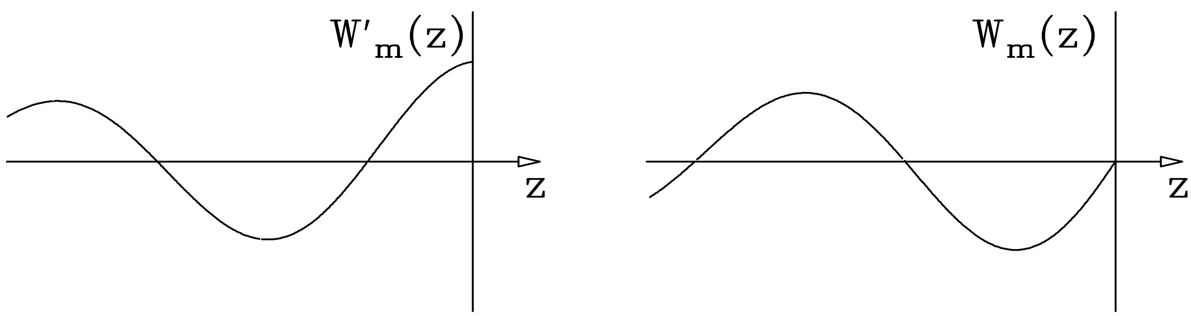

Figure 1.3: The longitudinal wake $W_m'(z)$ vanishes when $z > 0$ and is positive definite when $|z|$ is small. The transverse wake $W_m(z)$ starts out from zero and goes negative as $|z|$ increases.

a particle sees half of its own wake. For the transverse wake $W_m(z)$, it starts out from zero and goes negative as $|z|$ increases, as required by the P-W theorem. Thus, when the source particle is deflected, a transverse wake force is created in the direction that it will deflect particles immediately following in the *same* direction of the deflection of the source. Again, special attention should be placed to negative sign on the right side of the definition of $W_m(z)$ in Eq. (1.18). This implies that a particle will not see its own transverse wake at all. This leads to the important conclusion that a shorter bunch will be preferred if the transverse wake dominates, and a longer bunch will be preferred if the longitudinal wake dominates.

When $m = 0$ or the monopole, we have $\Delta p_\perp = 0$ while Δp_s is independent of (r, θ) and depends only on z. Thus particles in a thin transverse slice of the beam will see the same impulse in the s-direction according to the dependence of W_0' on z, as shown in Fig. 1.4. This impulse can lead to self-bunching or microwave instability.

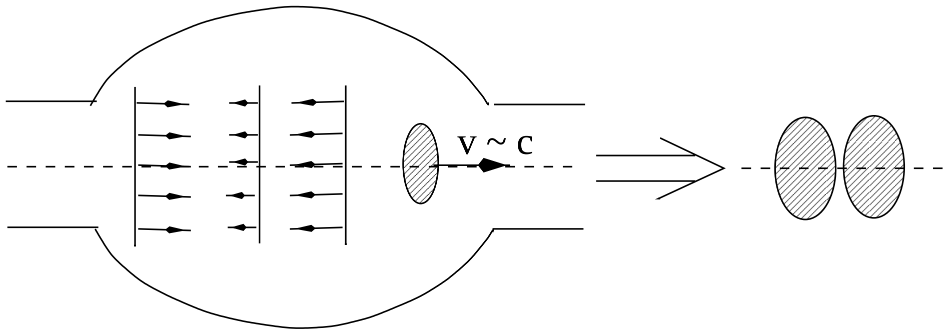

Figure 1.4: All particles in a vertical slice of the beam see exactly the same monopole wake impulse ($m = 0$) from the source according to the slice position z behind the source. This longitudinal variation of impulse effect on the slices can lead to longitudinal microwave instability.

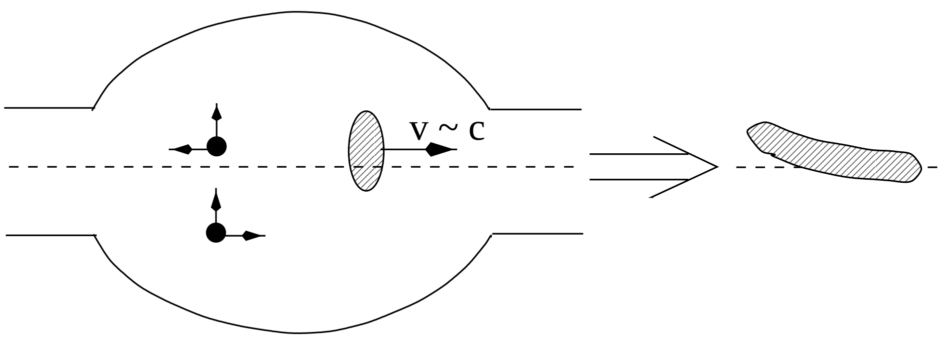

Figure 1.5: Kicks for all the particles in the slice from the dipole wake impulse also have the same magnitude; however, the longitudinal kicks are in opposite direction depending on whether the particles are above or below the axis of symmetry.

All particles in a vertical slice of the beam suffer exactly the same vertical kick from the dipole wake impulse ($m = 1$) which depends only on how far the slice is behind the dipole source. For the $m = 1$ or the dipole longitudinal we have from Eq. (1.18), $\Delta p_\perp$ independent of (r, θ) but depends on z only, while Δp_s is proportional to the offset in the x-direction, as shown in Fig. 1.5. Such an impulse can lead to beam breakup.

For the sake of convenience, many authors do not like to work with a negative z for the particles that are following. There is another convention that $W_m(z) = 0$ when $z < 0$. This does not change the physics and the direction of the wake forces will not be changed. Thus, instead of Fig. 1.3, we have Fig. 1.6 instead. A price has to be paid for this convention. We must interpret the connection between the longitudinal and transverse

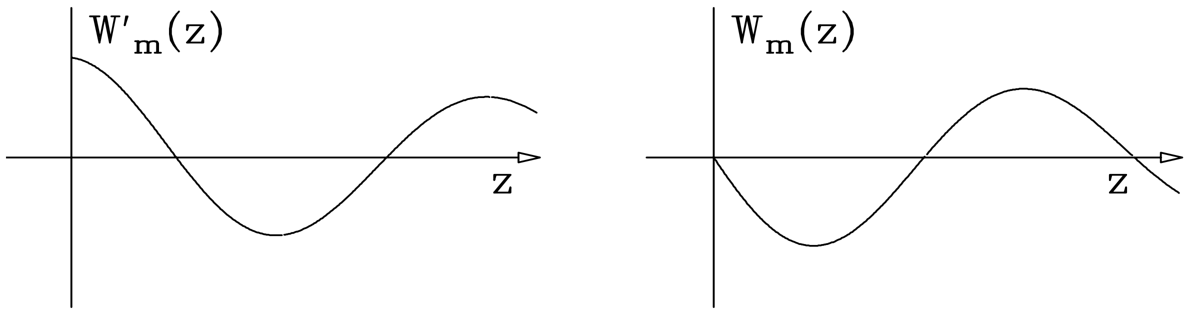

Figure 1.6: This is a different convention that the wake functions $W_m(z)$ vanish when $z<0$. Since the physics is the same, the wake functions are the same as in Fig. 1.3 and just the direction of z has been changed. In this convention, the interpretation $W'_m(z) \equiv -\dfrac{d}{dz}W_m(z)$ is required.

wakes as

$$W'_m(z) \equiv -\frac{d}{dz}W_m(z) \ . \tag{1.19}$$

This convention will be used for the rest of the lectures.†

1.5 COUPLING IMPEDANCES

Beam particles form a current, of which the component with frequency $\omega/(2\pi)$ is $I_0(s,t)=\hat{I}_0 e^{-i\omega(t-s/v)}$, where $\hat{I}_0$ may be complex. This current component at location s and time t will be affected by the wake of the preceding beam particles that pass the point s at time $t-z/v$ with the charge element $I_0(s,t-z/v)dz/v$. The total opposing voltage seen will be

$$V(s,t) = -\int_{-\infty}^{\infty} \hat{I}_0 e^{-i\omega[t-(s+z)/v]} W'_0(z)\frac{dz}{v} = -I_0(s,t)\int_{-\infty}^{\infty} e^{i\omega z/v} W'_0(z)\frac{dz}{v} \ . \tag{1.20}$$

Thus, we can identify the *longitudinal coupling impedance* of the vacuum chamber as

$$Z_0^{\|}(\omega) = \int_{-\infty}^{\infty} e^{i\omega z/v} W'_0(z)\frac{dz}{v} \ . \tag{1.21}$$

†The readers should be aware of yet another convention in the literature that the wake functions $W_m(z)$ and $W'_m(z)$ are defined in Eq. (1.18) *without* the negative signs on the right sides. As a result, the wake functions will have just the opposite signs of what are depicted in Fig. 1.6.

Similarly, when the current is displaced transversely by a, the opposing transverse force acting on a current particle is obtained by summing the charge element $I_0(s,t-z/v)dz/v$ passing s at time $t-z/v$,

$$\langle F_1^\perp(s,t)\rangle = -\frac{ea}{\ell}\int_{-\infty}^{\infty}\hat{I}_0 e^{-i\omega[t-(s+z)/v]}W_1(z)\frac{dz}{v} = -\frac{ea}{\ell}I_0(s,t)\int_{-\infty}^{\infty}e^{i\omega z/v}W_1(z)\frac{dz}{v}\ , \tag{1.22}$$

where $\langle F_1^\perp(s,t)\rangle$ is the transverse force average over a length ℓ covering the discontinuity of the vacuum chamber, and is therefore equal to $v\Delta p_\perp/\ell$, with $\Delta p_\perp$ being the transverse impulse studied in the previous sections. For an accelerator ring or storage ring, this length is taken to be the ring circumference C. We identify the *transverse coupling impedance* of the vacuum chamber as

$$Z_1^\perp(\omega) = \frac{i}{\beta}\int_{-\infty}^{\infty}e^{i\omega z/v}W_1(z)\frac{dz}{v}\ . \tag{1.23}$$

In both Eqs. (1.20) and (1.22), the lower limits of integration have been extended to $-\infty$, because the wake functions vanish when $z<0$. From Eq. (1.22), it is evident that we can also compute the transverse impedance by integrating the wake force averaged over one turn according to

$$Z_1^\perp(\omega) = -\frac{i}{e\beta I_0 a}\int_0^C F_1^\perp(s,t)\,ds\ , \tag{1.24}$$

where I_0a_1 represents the dipole source current. For $\mathcal{R}e\,Z_1^\perp(\omega)$, the force lags the displacement Ia_1 by $\frac{\pi}{2}$, hence the factor $-i$ in Eq. (1.24). The Lorentz factor $\beta=v/c$ is a convention.

Note that the longitudinal impedance is mostly the monopole ($m=0$) impedance and the transverse impedance is mostly the dipole ($m=1$) impedance, if the beam pipe cross section is close to circular and the particle path is close to the pipe axis. They have the dimensions of Ohms and Ohms/length, respectively. The impedances have the following properties:

1. $Z_0^\parallel(-\omega) = [Z_0^\parallel(\omega)]^*\ , \qquad Z_1^\perp(-\omega) = -[Z_1^\perp(\omega)]^*\ .$ (1.25)

2. $Z_0^\parallel(\omega)$ and $Z_1^\perp(\omega)$ are analytic with poles only in the lower half ω-plane. (1.26)

3. $Z_m^\parallel(\omega) = \dfrac{\omega}{c}Z_m^\perp(\omega)\ ,$ (1.27)

   for cylindrical geometry and each azimuthal harmonic $m \neq 0$.

4. $\mathcal{R}e\, Z_0^{\|}(\omega) \geq 0$, $\quad \mathcal{R}e\, Z_1^{\perp}(\omega) \geq 0$, $\quad$ when $\omega > 0$, (1.28)

5. $\displaystyle\int_0^\infty d\omega\, \mathcal{I}m\, Z_m^{\perp}(\omega) = 0$, $\quad \displaystyle\int_0^\infty d\omega\, \frac{\mathcal{I}m\, Z_m^{\|}(\omega)}{\omega} = 0$, (1.29)

   if the beam pipe has the same entrance and exit cross section.

The first follows because the wake functions are real, the second from the causality of the wake functions, and the third from the Panofsky-Wenzel theorem [1] between transverse and longitudinal electromagnetic forces. $\mathcal{R}e\, Z_m^{\|}(\omega) \geq 0$ is the result of the fact that the total energy of a particle or a bunch cannot be increased after passing through a section of the vacuum chamber where there is no accelerating external forces, while $\mathcal{R}e\, Z_m^{\perp}(\omega) \geq 0$ when $\omega > 0$ follows from the Panofsky-Wenzel theorem. The fifth property follows from that fact that $W_m(0) = 0$.

For a pure resistance R, the longitudinal wake is $W_0'(\omega) = R\delta(z/v)$. At low frequencies, the wall of the beam pipe is inductive. This wake function is $W_0'(\omega) = L\delta'(z/v)$, where L is the inductance.

For a nonrelativistic beam of radius a inside a circular beam pipe of radius b, the longitudinal space-charge impedance for $m = 0$ is[‡]

$$Z_0^{\|}(\omega) = i\frac{\omega}{\omega_0}\frac{Z_0}{2\gamma^2\beta}\left(1 + 2\ln\frac{b}{a}\right) , \tag{1.30}$$

where $Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 377\ \Omega$ is the impedance of free space, μ_0 and ϵ_0 are, respectively, the magnetic permeability and electric permitivity of free space, $\omega_0/(2\pi)$ is the revolution frequency of the beam particle with Lorentz factors γ and β. Although this impedance is capacitive, however, it appears in the form of a negative inductance. The corresponding wake function is

$$W_0'(z) = -\delta'(z/v)\frac{1}{\omega_0}\frac{Z_0}{2\gamma^2\beta}\left(1 + 2\ln\frac{b}{a}\right) . \tag{1.31}$$

The $m = 1$ transverse space-charge impedance for a length ℓ of the circular beam pipe is

$$Z_1^{\perp}(\omega) = i\frac{Z_0\ell}{2\pi\gamma^2\beta^2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right] , \tag{1.32}$$

[‡]Here, the space-charge force is seen by beam particles at the beam axis. If the force is averaged over the cross section of the beam, the first term in the brackets becomes $\frac{1}{2}$ instead of 1.

and the corresponding transverse wake function is

$$W_1(z) = \frac{Z_0 c\ell}{2\pi\gamma^2}\left[\frac{1}{a^2} - \frac{1}{b^2}\right]\delta(z)\ . \tag{1.33}$$

An important impedance is that of a resonant cavity. Near the resonant frequency $\omega_r/(2\pi)$, the longitudinal and transverse impedances can be derived from a RLC-parallel circuit:

$$Z_0^{\parallel}(\omega) = \frac{R_{0s}}{1 + iQ\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)}\ , \tag{1.34}$$

$$Z_1^{\perp}(\omega) = \frac{c}{\omega}\frac{R_{1s}}{1 + iQ\left(\dfrac{\omega_r}{\omega} - \dfrac{\omega}{\omega_r}\right)}\ . \tag{1.35}$$

Another example is the longitudinal impedance for a length ℓ of the resistive beam pipe:

$$Z_0^{\parallel}(\omega) = [1 - i\,\mathrm{sgn}(\omega)]\frac{\ell}{2\pi b\sigma_c\delta_{\mathrm{skin}}}\ , \tag{1.36}$$

where b is the radius of the cylindrical beam pipe, σ_c the conductivity of the pipe wall,

$$\delta_{\mathrm{skin}} = \sqrt{\frac{2c}{Z_0\mu_r\sigma_c|\omega|}}\ , \tag{1.37}$$

the skin depth at frequency $\omega/(2\pi)$, and μ_r the relative magnetic permeability of the pipe wall. The transverse impedance is

$$Z_1^{\perp}(\omega) = [1 - i\,\mathrm{sgn}(\omega)]\frac{\ell c}{\pi\omega b^3\sigma_c\delta_{\mathrm{skin}}}\ , \tag{1.38}$$

and is related to the longitudinal impedance by

$$Z_1^{\perp}(\omega) = \frac{2c}{b^2\omega}Z_0^{\parallel}(\omega)\ . \tag{1.39}$$

The above relation has been used very often to estimate the transverse impedance from the longitudinal. However, we should be aware that this relation holds only for resistive impedances of a cylindrical beam pipe. The monopole longitudinal impedance and the dipole transverse impedance belong to different azimuthals; therefore they should not be related.

More expressions for impedances resulting from various types of discontinuity can be found in the *Handbook of Accelerator Physics and Engineering* [3].

1.6 EXERCISES

1.1. Prove the properties of the impedances in Eqs. (1.25)-(1.28).

1.2. Using a RLC-parallel circuit, derive the longitudinal impedance in Eq. (1.34) by identifying $R_{0s} = R$, $\omega_r = 1/\sqrt{LC}$, and $Q = R\sqrt{C/L}$. Then show that the wake function is $W_0' = 0$ for $z < 0$, and for $z > 0$,

$$W_0'(z) = \frac{\omega_r R_{0s}}{Q} e^{-\alpha z/v} \left[\cos\frac{\bar{\omega} z}{v} - \frac{\alpha}{\bar{\omega}} \sin\frac{\bar{\omega} z}{v} \right] \ , \tag{1.40}$$

with $\alpha = \omega_r/(2Q)$ and $\bar{\omega} = \sqrt{\omega_r^2 - \alpha^2}$. Similarly, show that

$$W_1(z) = -\frac{R_{1s} c\omega_r}{Q\bar{\omega}_r} e^{-\alpha z/v} \sin\frac{\bar{\omega} z}{v} \ , \tag{1.41}$$

for $z > 0$ and zero otherwise.

1.3. Show that the wake functions corresponding to the longitudinal resistive wall impedance of Eq. (1.36) and the transverse resistive wall impedance of Eq. (1.38) for a length ℓ are, respectively,

$$W_0'(z) = -\frac{\beta^{3/2} c\ell}{4\pi b z^{3/2}} \sqrt{\frac{Z_0 \mu_r}{\pi \sigma_c}} \ , \tag{1.42}$$

$$W_1(z) = -\frac{\beta^{3/2} c\ell}{\pi b^3 z^{1/2}} \sqrt{\frac{Z_0 \mu_r}{\pi \sigma_c}} \ , \tag{1.43}$$

where b is the beam pipe radius, σ_c is the conductivity and μ_r the relative magnetic permeability of the beam pipe walls. The above are only approximates and are valid for $b\chi^{1/3} \ll z \ll b/\chi$, where $\chi = 1/(b\sigma_c Z_0)$. When $z \ll b\chi^{1/3}$, $W_0'(z)$ should have the proper positive sign.

Bibliography

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